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ON THE METHOD OF THE LOCAL POTENTIAL AS APPLIED TO THE SOLUTION OF THE EQUATIONS OF DIFFUSION

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Abstract—The method of the local potential, viewed as a tool for the approximate solution of the equations governing the diffusion of heat and vorticity, is examined critically.

Conditions are derived which are sufficient to establish an exact correspondence between local potential and multi-moment solutions of the equations of diffusion; no local potential formulation of the equations of diffusion has been found which does not satisfy the given conditions.

NOMENCLATURE

a_i, a_{ij}, functions occurring in trial solutions;
c, heat capacity per unit volume;
c_i, concentration of solutes in multicomponent mixture;
D_{ij}, diffusion co-efficients;
f, function;
F, function;
g, g_i, functions;
G, G_i, functions;
H, function;
I, I_j, functionals;
L, linear differential operator;
l, length;
p, pressure;
$\mathbf{q}$, thermal current density;
q_i, functions of time;
R, region of definition of an equation;
t, time;
u, function, which is streamwise velocity component in boundary layer flow;
U_e, velocity at "edge" of boundary layer;
v, normal component of velocity in boundary layer;
x, y, independent variables;
Y, independent variable;
α, diffusion distance;
$\bar{\alpha}$, a constant;
γ, defined in section 6 (see equation 6.3);
Γ, boundary of region within which equations are defined;
δ, diffusion distance or variational symbol;
δ_i, diffusion distances;
ζ, η, independent variables;
θ, temperature;
κ, thermal conductivity;
μ, viscosity;
ν, kinematic viscosity;
ρ, ρ_i, densities;
τ, relaxation time;
ϕ, transformed dependent variable;
ϕ_i, approximating functions;
Φ, Φ_i, functions.

Subscript

0, refers to dependent variables not subject to variation.

Superscripts

0, s, refer to dependent variables not subject to variation.

INTRODUCTION

AS A TOOL for the approximate solution of the equations of diffusion and mass transfer, the method of the local potential presents a somewhat controversial image. That the technique is related to the Galerkin technique for the solution of differential equations [1] is accepted beyond question (indeed, it has been claimed [2] that, "the so called self consistent approximation based on the local potential *is* the Galerkin scheme") but why there is a relation and precisely what that relation is has yet to be made clear; the appearance of a number of recent papers, [3–6] employing the method of the local potential, billed as a new variational technique, is ample proof of this statement.

The Galerkin technique owes its origins to the study of the classical linear elliptic equations. The solution of such equations is well known to be equivalent to the problem of finding that function, of a given class of functions, which will render a given functional stationary [1, 7]. The Galerkin approach, although

equivalent to the variational technique of Ritz [8] in those cases where an equivalent variational formulation of the problem exists, is not, however, dependent on a variational formulation and it is for this reason that it has gained its prominent position as an approximation technique.

To solve the equation

$$L(u) = f, \tag{1.1}$$

subject to

$$u = 0 \text{ on } \Gamma, \tag{1.2}$$

where Γ is the boundary of the region R in which the equation is defined, the Galerkin approach sets

$$u(x, y) = \sum_{i=1}^{n} a_i \phi_i(x, y), \tag{1.3}$$

where the functions ϕ_i are the first n members of a complete set of functions, each member of which satisfies condition (1.2). To determine the n unknowns, $a_1, \ldots, a_n$, the residual which results when (1.3) is substituted in 1.1 is made orthogonal to each of the functions $\phi_i, i = 1, \ldots, n$; thus the a_i are determined from the n equations

$$\iint_R \left\{ L\left(\sum_{i=1}^{n} a_i \phi_i \right) - f \right\} \phi_j \, dR = 0, \quad j = 1, 2, \ldots, n.$$

In the limit this procedure will provide an exact solution to the problem, since the only function orthogonal to every member of a complete set of functions is the null function. In principle, the technique is applicable to a much wider range of problems than is indicated in the above example. In cases where the approximating functions ϕ_i do not satisfy some or all of the imposed boundary conditions a residual will result when (1.3) is substituted in (1.2). The procedure in such cases is to add the boundary residual to the differential equation residual and to then force the resulting residual to be orthogonal to each of the functions $\phi_i, i = 1, 2, \ldots, n$ [2, 9]. In general, it is unusual for this procedure to be followed and in both the Galerkin and local potential schemes the situation in which the approximating functions satisfy the same boundary and initial conditions as the sought solution is by far the more common one.

The practical difficulties which confront the Galerkin approach as applied to the solution of the equations of diffusion can perhaps best be illustrated by considering a specific example, taken here to be the steady two dimensional incompressible boundary layer equations. No variational principle exists for this equation [10] and any procedure based on the Galerkin scheme can only be interpreted in the sense of forcing a residual to be orthogonal to the first n members of a complete set of functions.

With the streamwise velocity component u approximated in the form

$$u(x, y) = \sum_{i=1}^{n} a_i(x) \phi_i(x, y)^*$$

the most basic requirements on the set of approximating functions ϕ_i are that

$$\frac{\partial \phi_i}{\partial y} \to 0, \quad y \to \infty \quad \text{and} \quad \phi_i(x, 0) = 0;$$

in addition u must satisfy the condition

$$u(x, y) \to U_e(x), \quad y \to \infty,$$

where U_e denotes the main stream velocity.

Two direct applications of the Galerkin approach to this problem have been developed. Bossel [11] guided by the exact Falkner Skan solutions, has made use of a transformation of variables of the form

$$\zeta = x, \quad Y = \frac{y}{g(x)},$$

where g is a specified function of x which depends on the nature of the mainstream velocity, U_e; the approximating functions ϕ_i are taken as

$$\phi_i = \phi_i(\eta) = \eta^i (1 - \eta),$$

where

$$\eta = \exp(-\bar{\alpha} Y), \quad \bar{\alpha} \text{ a constant},$$

so that

$$u(\zeta, \eta) = \sum_{i=0}^{n} a_i(\zeta) \eta^i (1 - \eta),$$

with

$$a_0(\zeta) = U_e(\zeta).$$

Since

$$\frac{\partial}{\partial y} = \frac{-\bar{\alpha}}{g(x)} \eta \frac{\partial}{\partial \eta},$$

u approaches U_e in an exponential manner [although, as remarked by Bossel, a more appropriate representation would be $\eta = \exp(-y^2 \bar{\alpha}/g(x))$].

MacDonald [12] has employed a transformation in which the new dependent variable is the square of the shear stress and the independent variables are x and u, where u denotes the streamwise velocity component. Further elementary transformations of the dependent

*In what follows it will be assumed that all variables have been non dimensionalised in the standard manner. Thus the boundary layer equation is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{2} \frac{d}{dx} U_e^2 + \frac{\partial^2 u}{\partial y^2}.$$

variable result in an equation with dependent variable ϕ, which must satisfy the conditions:

$$\phi = 0,\ u = 1;\quad \frac{\partial \phi}{\partial u} = 0,\ u = 0.$$

Various complete sets, a typical one of which is

$$\phi_k(u) = (1-u^3)u^{k+1},\quad k = -1, 2, 3, \ldots,$$

are then employed.

Yet a third and, at a first view, more obvious approach is to introduce a boundary layer thickness $\delta(x)$, i.e. a distance normal to the boundary within which the streamwise velocity changes from the value zero, at the wall, to within a small percentage of the value $U_e(x)$ in the mainstream. A change of variables

$$\zeta = x,\quad \eta = \frac{y}{\delta(x)}$$

then allows an approximation of the form

$$u(\zeta, \eta) = \sum_{i=1}^{n} a_i(\zeta)\phi_i(\eta), \tag{1.4}$$

where

$$\phi_i(0) = 0,\ \frac{\partial \phi_i}{\partial \eta} = 0,\ \eta = 1,\ i = 1, 2, \ldots, n,$$

and

$$\sum_{i=1}^{n} a_i(\zeta)\phi_i(1) = U_e(\zeta).$$

This is the approach that has been used in the local potential formulations of Schechter [13, 14] and Weihs and Gal-Or [3, 4].

The essence of the local potential technique is the reformulation of the original set of partial differential equations in terms of a restricted variational principle; this principle, in conjunction with the approximation (1.4), is used to generate a set of ordinary differential equations for the $a_i(\zeta)$, $i = 1, 2, \ldots, n$, and $\delta(\zeta)$. It will be shown that subject to certain conditions being satisfied such procedures are equivalent to determining the a_i by the Galerkin scheme and determining δ by a variation of the well known Kármán–Pohlhausen approach [15]. A number of examples of this equivalence will be presented.

2. THE METHOD OF THE LOCAL POTENTIAL

Restricted variational principles are characterised by the appearance within the functional to be made stationary of the actual function which renders that functional stationary. In the variational analysis leading from the restricted principle to the corresponding Euler–Lagrange equation this latter function is not subjected to variation. As a simple example, the problem of finding the solution to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \tag{2.1}$$

$$u = 0 \text{ on } \Gamma, \tag{2.2}$$

is equivalent to finding that function u, of the class C of functions satisfying (2.2), which is such that

$$\delta \iint_R \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 2fu \right\} \mathrm{d}x\, \mathrm{d}y \equiv \delta I_1 = 0. \tag{2.3}$$

One of the restricted variational formulations of the same problem is that of finding that function u of the class C which is such that

$$\delta \iint_R u \left\{ \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} - f \right\} \mathrm{d}x\, \mathrm{d}y \equiv \delta I_2 = 0; \tag{2.4}$$

in equation (2.4) the expression within brackets is to be evaluated at the stationary state, denoted by suffix zero, and is therefore not subjected to variation.

If an approximation to u is now sought in the form (1.3) then, in the manner of Ritz [8] the a_i can, in the case of (2.3), be determined from the set of equations

$$\frac{\mathrm{d}I_1}{\mathrm{d}a_i} = 0,\quad i = 1, 2, \ldots, n, \tag{2.5}$$

or, in the case of (2.4), from the set

$$\frac{\mathrm{d}I_2}{\mathrm{d}a_i} = 0,\quad i = 1, 2, \ldots, n. \tag{2.6}$$

In the case of (2.6) the procedure is to regard suffix zero terms in (2.4) as invariant, to differentiate with respect to a_i, $i = 1, 2, \ldots, n$, and to subsequently set

$$u_0(x, y) = \sum_{i=1}^{n} a_i(x)\phi_i(x, y).$$

Mathematically, the latter approach is identical to that used in the method of the local potential. It is easy to see that provided each of the ϕ_i satisfy condition (2.2) an identical set of algebraic equations for the a_i is obtained from equations (2.5). A further restricted principle can be obtained by application of the divergence theorem to (2.4); this yields, after use of (2.2),

$$\delta \iint_R \{\nabla u . \nabla u_0 + uf\}\, \mathrm{d}R = 0. \tag{2.7}$$

Alternatively, (2.7) may be obtained from the Euler–Lagrange equation,

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0,$$

corresponding to the principle

$$\delta \iint_R F(u, u_x, u_y)\, dR = 0.$$

From the above discussion it is clear that for the simple elliptic problem described by (2.1) and (2.2) there are at least two restricted variational formulations which lead to identical sets of equations for the a_i in (1.3); furthermore, this set of equations is identical to that obtained via the classical Ritz procedure or, for that matter, the classical Galerkin procedure.

3. RESTRICTED PRINCIPLES AND THE EQUATIONS OF DIFFUSION

The applications of the method of the local potential to the solution of the equations of diffusion have invariably introduced a scaled co-ordinate $\eta = y/\delta(x)$, where δ is some penetration distance associated with a diffusing quantity which may, for example, be vorticity, heat or concentration of solute [3–6, 13, 14]. The solution to two-dimensional diffusion problems is therefore approximated in the form

$$u(x, y) = \sum_{i=1}^{n} a_i(x)\phi_i(x, \eta), \tag{3.1}$$

where the ϕ_i, $i = 1, 2, \ldots, n$, are chosen to satisfy the basic conditions of the problem (an obvious condition on the ϕ_i being $(\partial\phi_i/\partial\eta) \to 0, \eta \to 1$). A variety of methods may be employed to determine δ. The local potential approach is to determine the $(n+1)$ unknowns in (3.1) in the manner of Ritz [8]; for the a_i, $i = 1, \ldots, n$, this procedure is equivalent to the Galerkin scheme, but this clearly cannot be so for the remaining unknown, δ.

Consider the general problem of solving the equation

$$F(u, u_x, u_y, u_{xx}, u_{yy}) = 0, \tag{3.2}$$

which may be nonlinear in form. A restricted variational principle for equation (3.2) is that of finding that function u of some class C which will render stationary the functional

$$I = \iint_R uF\left(u_0, \frac{\partial u_0}{\partial x}, \frac{\partial u_0}{\partial y}, \frac{\partial^2 u_0}{\partial x^2}, \frac{\partial^2 u_0}{\partial y^2}\right) dR \equiv \iint_R uF_0\, dR. \tag{3.3}$$

Suppose next that a solution is sought in the form (3.1), where the functions ϕ_i are members of the class C. Then

$$I = \iint_R \left[\sum_{i=1}^{n} a_i\phi_i\right] F_0\, dR$$

and the equations

$$\frac{\partial I}{\partial a_i} = 0, \quad i = 1, 2, \ldots, n,$$

yield a set of equations for the a_i which are identical to the set obtained when the a_i in (3.1) are determined by the Galerkin approach. The remaining equation,

$$\frac{\partial I}{\partial \delta} = 0,$$

gives

$$\iint_R \left(\sum_{i=1}^{n} a_i \frac{\partial \phi_i}{\partial \delta}\right) F_0\, dR = 0^*. \tag{3.4}$$

But

$$\frac{\partial \phi_i}{\partial \delta} = \frac{-\eta}{\delta}\frac{\partial \phi_i}{\partial \eta} = \frac{-y}{\delta}\frac{\partial \phi_i}{\partial y}.$$

Equation (3.4) may now, in the spirit of the local potential, be written in the form

$$\iint_R y \frac{\partial u}{\partial y} F\, dR = 0. \tag{3.5}$$

4. EQUIVALENCE OF RESTRICTED PRINCIPLES

We consider at this stage the class of variational principles

$$\delta \int_0^l dx \int_0^{\alpha(x)} F(u, u_x, u_y)\, dy = 0, \tag{4.1}$$

where l is a constant and $\alpha(x)$ is some unknown penetration distance associated with some diffusing quantity. We assume that approximations for u are to be sought in the form

$$u = \sum_{i=1}^{n} a_i(x)\phi_i\left(\frac{y}{\alpha(x)}\right). \tag{4.2}$$

Substitution of (4.2) in (4.1) gives

$$\delta \int_0^l dx\, \Phi(\alpha, \alpha', a_1, \ldots, a_n, a_1', \ldots, a_n') = 0, \tag{4.3}$$

where a prime denotes differentiation with respect to x and

$$\Phi = \int_0^{\alpha(x)} F(u, u_x, u_y)\, dy. \tag{4.4}$$

The Euler–Lagrange equations corresponding to (4.3) are

$$\frac{\partial \Phi}{\partial \alpha} - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial \alpha'}\right) = 0,$$

$$\frac{\partial \Phi}{\partial a_j} - \frac{d}{dx}\left(\frac{\partial \Phi}{\partial a_j'}\right) = 0, \quad j = 1, 2, \ldots, n,$$

*Note that $F_0)_{y=\delta(x)} = 0$; those cases where part of the boundary of R is the curve $y = \delta(x)$ are thus accounted for in (3.4).

so that, from (4.4),

$$\int_0^{\alpha}\left[\frac{\partial F}{\partial \alpha}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \alpha'}\right)\right]\mathrm{d}y=0, \tag{4.5}$$

$$\int_0^{\alpha}\left[\frac{\partial F}{\partial a_j}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial a_j'}\right)\right]\mathrm{d}y=0, \quad j=1,2,\ldots,n, \tag{4.6}$$

provided that

$$F(u,u_x,u_y)|_{y=\alpha}=0,$$

$$\left.\frac{\partial F}{\partial \alpha'}\right|_{y=\alpha}=0,$$

and

$$\left.\frac{\partial F}{\partial a_j'}\right|_{y=\alpha}=0, \quad j=1,2,\ldots,n.$$

Now

$$\frac{\partial F}{\partial a_j}=\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial a_j}+\frac{\partial F}{\partial u_x}\frac{\partial u_x}{\partial a_j}+\frac{\partial F}{\partial u_y}\frac{\partial u_y}{\partial a_j}$$
$$=\phi_j\frac{\partial F}{\partial u}-\frac{y}{\alpha^2}\alpha'\phi_j'\frac{\partial F}{\partial u_x}+\frac{1}{\alpha}\phi_j'\frac{\partial F}{\partial u_y},$$

where

$$\phi_j'=\frac{\mathrm{d}\phi_j}{\mathrm{d}\eta}, \quad \eta=\frac{y}{\alpha(x)}.$$

Similarly,

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial a_j'}\right)=\frac{\partial}{\partial x}\left(\phi_j\frac{\partial F}{\partial u_x}\right)=\frac{-y}{\alpha^2}\alpha'\phi_j'\frac{\partial F}{\partial u_x}+\phi_j\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right).$$

Equation (4.6) now gives

$$\int_0^{\alpha}\left[\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\phi_j\,\mathrm{d}y=0, \quad j=1,\ldots,n,$$

provided that

$$\phi_j\left.\frac{\partial F}{\partial u_y}\right|_0^{\alpha}=0, \quad j=1,\ldots,n.$$

In the same way we can show that (4.5) reduces to

$$\int_0^{\alpha}\left[\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\right]\left[\frac{-y}{\alpha^2}\sum_{i=1}^{n}a_i\phi_i'\right]\mathrm{d}y$$
$$+\int_0^{\alpha}\frac{\partial F}{\partial u_y}\frac{\partial u_y}{\partial \alpha}\,\mathrm{d}y=0.$$

But

$$\frac{\partial u_y}{\partial \alpha}=\frac{-y}{\alpha}\frac{\partial u_y}{\partial y}-\frac{1}{\alpha}\frac{\partial u}{\partial y},$$

so that

$$\int_0^{\alpha}\frac{\partial F}{\partial u_y}\frac{\partial u_y}{\partial \alpha}\,\mathrm{d}y=\int_0^{\alpha}\frac{y}{\alpha}u_y\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\mathrm{d}y-u_y\left.\frac{\partial F}{\partial u_y}\right|_{y=\alpha}.$$

Thus

$$\int_0^{\alpha}\left[\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]yu_y\,\mathrm{d}y=0,$$

provided that

$$u_y\left.\frac{\partial F}{\partial u_y}\right|_{y=\alpha}=0.$$

The theory given above and in section 3 may now be summarised as follows:

Theorem 1

If two variational principles,

$$\delta\int_0^{l}\mathrm{d}x\int_0^{\alpha(x)}G_i(u,u_x,u_y)\,\mathrm{d}y=0, \quad i=1,2, \tag{4.7}$$

can be derived from the same partial differential equation,

$$H(u,u_x,u_y,u_{xx},u_{yy})=0, \quad \text{say,}$$

and if a solution for u is sought in the form

$$u(x,y)=\sum_{i=1}^{n}a_i(x)\phi_i\left(\frac{y}{\alpha(x)}\right)$$

then provided that for $i=1,2$, $j=1,2,\ldots,n$, the expressions

$$\left.G_i\right|_{y=\alpha}, \quad \left.\frac{\partial G_i}{\partial u_x}\cdot\frac{\partial u_x}{\partial \alpha'}\right|_{y=\alpha}, \quad \left.\frac{\partial G_i}{\partial u_x}\cdot\frac{\partial u_x}{\partial a_j'}\right|_{y=\alpha},$$
$$\left.\phi_j\frac{\partial G_i}{\partial u_y}\right|_{y=0}^{y=\alpha}, \quad \left.u_y\frac{\partial G_i}{\partial u_y}\right|_{y=\alpha}$$

are independent of $\alpha, a_i, i=1,2,\ldots,n$, the sets of ordinary differential equations for the determination of the a_i, $i=1,2,\ldots,n$, and α will be identical and will, respectively, be the set obtained from

$$\int_0^{\alpha}\phi_i H\,\mathrm{d}y=0, \quad i=1,2,\ldots,n,$$

and

$$\int_0^{\alpha}y\frac{\partial u}{\partial y}H\,\mathrm{d}y=0. \tag{4.8}$$

Definition

When, in an approximation scheme for the solution of parabolic partial differential equations, two variational principles, which can be derived from the same partial differential equation (or equations), lead to identical sets of ordinary differential equations for the unknown parameters describing the solution they will be said to be equivalent.

Theorem 2

In those cases where the upper limit α in the variational principles (4.7) is replaced by ∞ and a solution is sought in the form (4.2) sufficient conditions

for the equivalence of the principles are that for $i = 1, 2$, $j = 1, 2, \ldots, n$, the expressions

$$\phi_j \frac{\partial G_i}{\partial u_y}\bigg|_{y=0}^{y=\infty} \quad \text{and} \quad u_y \frac{\partial G_i}{\partial u_y}\bigg|_{y=\infty}$$

are independent of α, a_j, $j = 1, 2, \ldots, n$.

5. REMARKS ON PREVIOUS LOCAL POTENTIAL FORMULATIONS

Schechter [13, 14] has applied the local potential technique to the solution of the equations governing the incompressible flow in a steady two dimensional boundary layer. The derived functional (see equation (15) of [14]) is

$$F_1 = \int_0^L \mathrm{d}x \int_0^\infty \left[-u_0 v_0 \frac{\partial u}{\partial y} + 2u_0 \frac{\partial u_0}{\partial x} u + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 \right] \mathrm{d}y$$

and a solution for u is sought in the form

$$u = \mathrm{erf}\left(\frac{y}{\delta(x)}\right). \tag{5.1}$$

From Theorem 2 the equivalence of F_1 and F_2 where

$$F_2 = \int_0^L \mathrm{d}x \int_0^\infty \left[u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} - \frac{\partial^2 u_0}{\partial y^2} \right] u \, \mathrm{d}y, \tag{5.2}$$

follows immediately; thus* the equation for $\delta(x)$ is

$$\int_0^\infty \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} \right] y \frac{\partial u}{\partial y} \mathrm{d}y = 0, \tag{5.3}$$

where u is given by (5.1) and v follows from the mass conservation equation.

The work of Weihs and Gal-Or [3] concerns boundary layer flows which include viscous effects, coupled heat and multi-component mass transfer and chemical reactions. The approach separates a local potential, derived from the equations of motion, into two parts; these parts separately represent the transport of momentum and the transport of heat and mass.

In the case of uniform flow of a homogeneous incompressible fluid over a flat plate the derived functional reduces to

$$I_1 = \int_0^L \mathrm{d}x \int_0^{\delta(x)} \left[-2u_0 \frac{\partial u_0}{\partial x} u + \left(u_0 v_0 - \frac{\partial u_0}{\partial y} \right) \frac{\partial u}{\partial y} \right] \mathrm{d}y. \tag{5.4}$$

A solution for u is sought in the form $u = \phi(\eta)$, where $\eta = y/\delta$, $\phi(1) = 1$ and $\phi'(1) = 0$; it follows from (5.4) and Theorem 1 that I_1 is equivalent to

$$I_2 = \int_0^L \mathrm{d}x \int_0^{\delta(x)} \left[u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} - \frac{\partial^2 u_0}{\partial y^2} \right] u \, \mathrm{d}y.\dagger$$

*Since F_1 and F_2 are both derived from the same differential equation (see [14]).

†Equation 27 of [3] should read

$$\int_0^L \left[\rho U^3 \frac{(f^s)'}{(f^s)^3} 0{\cdot}0291667 + \mu U^2 0{\cdot}2666667 \right] \mathrm{d}x = 0.$$

The second example given in [3] concerns the isothermal multi-component boundary layer on a flat plate lying parallel to a uniform stream. The functionals quoted for this problem are (5.4) and the functional

$$I_3 = \iint_R \left[\sum_{i=1}^{n-1} \left\{ -\frac{\partial}{\partial x}(u_0 c_i^0) c_i + v_0 c_i^0 \frac{\partial c_i}{\partial y} - \sum_{j=1}^{n-1} D_{ij} \frac{\partial c_j^0}{\partial y} \frac{\partial c_i}{\partial y} \right\} \right] \mathrm{d}y \, \mathrm{d}x, \tag{5.5}$$

where

$$\rho = \sum_{i=1}^{n} \rho_i, \quad c_i = \rho_i/\rho, \quad i = 1, 2, \ldots, (n-1),$$

n is the number of chemical components present in the fluid and R denotes a rectangular control volume with sides parallel to the co-ordinate axes. Solutions for the c_i are sought in the form

$$c_i = \sum_{j=1}^{m} a_{ij}(x) \phi_j\left(\frac{y}{\delta_i(x)}\right), \quad i = 1, 2, \ldots, (n-1). \tag{5.6}$$

From (5.5), (5.6) and Theorem 2 it is not difficult to deduce that the functionals I_3 and

$$I_4 = \iint_R \left\{ \sum_{i=1}^{n-1} \left[u_0 \frac{\partial c_i^0}{\partial x} + v_0 \frac{\partial c_i^0}{\partial y} - \frac{\partial}{\partial y} \sum_{j=1}^{n-1} D_{ij} \frac{\partial c_j^0}{\partial y} \right] c_i \right\} \mathrm{d}y \, \mathrm{d}x$$

are equivalent provided that for all $i = 1, 2, \ldots, (n-1)$ and $j = 1, 2, \ldots, m$ the expressions

$$\phi_j \left(v_0 c_i^0 - \sum_{j=1}^{n-1} D_{ij} \frac{\partial c_j^0}{\partial y} \right)\bigg|_0^\infty$$

and

$$\frac{\partial c_i}{\partial y} \left(v_0 c_i^0 - \sum_{j=1}^{n-1} D_{ij} \frac{\partial c_j^0}{\partial y} \right)\bigg|_{y=\infty}$$

are independent of a_{ij} and δ_l. Use of the representations employed in [3] for the c_i shows that the above conditions are satisfied.

Further applications of the above theory may be found by reference to the papers of Yu [5] and Lemieux *et al.* [6].

We note that in view of the close association of equation (4.8) with the established integral methods of fluid mechanics [15] it is to be expected that the local potential formulation of problems involving adverse pressure gradients will lead to unreliable results.

6. THE VARIATIONAL PRINCIPLE OF VUJANOVIC

Vujanovic [18] has recently presented an interesting variational principle for the dissipative wave equation

$$\tau \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} - \frac{\kappa}{c} \nabla^2 \theta = 0, \tag{6.1}$$

where θ denotes temperature, t the time, c the heat capacity per unit volume, κ the thermal conductivity and τ a relaxation time, which is related to the thermal current density $\mathbf{q}$ by the equation

$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\kappa \nabla \theta. \tag{6.2}$$

For a discussion of equations (6.1) and (6.2) the reader is referred to [19].

The one dimensional version of Vujanovic's principle is that

$$\delta \int_{t_0}^{t_1} \mathrm{d}t \int_{y_0}^{y_1} \left[\tau \left(\frac{\partial \theta}{\partial t} \right)^2 - \gamma \left(\frac{\partial \theta}{\partial y} \right)^2 \right] \mathrm{e}^{t/\tau} \, \mathrm{d}y = 0, \tag{6.3}$$

where

$$\gamma = \frac{\kappa}{c}.$$

In the derivation leading from (6.3) to the one dimensional version of (6.1) it is assumed that the temperature variation $\delta\theta$ vanishes on the boundaries. Since the classical diffusion equation is obtained by taking the limit $\tau \to 0$ in (6.1), the procedure with regard to (6.3) is to perform the variation, divide by $\mathrm{e}^{t/\tau}$ and subsequently take the limit $\tau \to 0$. Applications of the principle are presented in [18] and [20]; apart from one example, where the results are identical to those obtained by use of the Galerkin approach, the method is to introduce a scaled co-ordinate $\eta = y/\delta(t)$ and to search for a solution

$$\theta = \theta(\eta).$$

Consider now

$$I = \int_0^{t_1} \Phi(t, \delta, \delta') \, \mathrm{d}t,^*$$

where

$$\Phi = \int_0^{\delta(t)} \left[\tau \left(\frac{\partial \theta}{\partial t} \right)^2 - \gamma \left(\frac{\partial \theta}{\partial y} \right)^2 \right] \mathrm{e}^{t/\tau} \, \mathrm{d}y. \tag{6.4}$$

Then $\delta I = 0$ gives the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \Phi}{\partial \delta'} \right) - \frac{\partial \Phi}{\partial \delta} = 0;$$

use of equation (6.4), together with relations like

$$\frac{\partial \theta}{\partial y} = \frac{1}{\delta} \theta', \quad \frac{\partial \theta}{\partial t} = \frac{-y}{\delta^2} \delta' \theta',$$

now establishes, after elementary manipulation, that

$$\int_0^{\delta} \left\{ \tau \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \theta}{\partial x} - \gamma \frac{\partial^2 \theta}{\partial y^2} \right\} \frac{y}{\delta} \frac{\partial \theta}{\partial y} \, \mathrm{d}y - \left(\frac{\theta'}{\delta} \right)_{y=\delta}^2 \left[\frac{\tau}{2} (\delta')^2 - \frac{\gamma}{2} \right] = 0. \tag{6.5}$$

The assumption that the variation $\delta\theta = 0$ on the boundaries is equivalent in part to the statement that all possible trial functions must satisfy the same condition on $y = \delta(t)$; since $\delta(t)$ is defined to be a penetration distance $\theta'(\eta)|_{y=\delta} = 0$ and (6.5) accordingly reduces to

$$\int_0^{\delta} \left\{ \tau \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial \theta}{\partial x} - \gamma \frac{\partial^2 \theta}{\partial y^2} \right\} y \frac{\partial \theta}{\partial y} \, \mathrm{d}y = 0. \tag{6.6}$$

Equation (6.6) may therefore be used as an alternative to (6.3) in any example in which $\theta = \theta(y/\delta(t))$. This conclusion is readily demonstrated when applied to the examples contained in [18] and [20]. In the limit $\tau \to 0$ (6.6) gives

$$\int_0^{\delta} \left[\frac{\partial \theta}{\partial x} - \gamma \frac{\partial^2 \theta}{\partial y^2} \right] y \frac{\partial \theta}{\partial y} \, \mathrm{d}y = 0$$

as the equation for the determination of δ; but it is not at all clear that the solution for $\theta(y/\delta)$ obtained in this way is any more reliable than that obtained in some other way, such as, for example, by solution of

$$\int_0^{\delta} \left[\frac{\partial \theta}{\partial x} - \gamma \frac{\partial^2 \theta}{\partial y^2} \right] \mathrm{d}y = 0.$$

(See, in this connection, [21].)

CONCLUSION

This paper has examined the method of the local potential from a critical viewpoint. Conditions have been presented which when satisfied identify the method with a much simpler, more readily comprehensible approach. No application of the local potential technique has been found which fails to satisfy such conditions. The method cannot, therefore, be viewed as a viable new technique for the solution of the equations of diffusion, but must be regarded as a method producing results of a comparable standard to those obtained by application of the established Kármán–Pohlhausen type integral techniques of fluid mechanics [15].

*In what follows a prime will denote differentation with respect to t in the case of $\delta(t)$ and with respect to η in the case of $\theta(\eta)$.

REFERENCES

1. L. V. Kantorovich and V. I. Krylov, *Approximate Methods of Higher Analysis*, 3rd ed., Ch. IV. Netherlands Publishing Co., Groningen (1964).
2. B. A. Finlayson and L. E. Scriven, On the search for variational principles, *Int. J. Heat Mass Transfer* **10**, 799–821 (1967).
3. D. Weihs and B. Gal-Or, General variational analysis of hydrodynamic, thermal and diffusional boundary layers, *Int. J. Engng Sci.* **8**, 231–249 (1970).
4. B. Gal-Or and D. Weihs, Variational analysis of high mass transfer rates from spherical particles boundary

layer injection suction considerations at low particle Reynolds numbers and high Peclet numbers, *Int. J. Heat Mass Transfer* **15**, 2027–2044 (1972).
5. J. C. M. Yu, A new variational principle, *Q. Jl Mech. Appl. Math.* **XXV**, 265–272 (1972).
6. P. F. Lemieux, R. N. Dubey and T. E. Unny, Variational method for a pseudoplastic fluid in a laminar boundary layer over a flat plate, *J. Appl. Mech.* **38**, 345–350 (1971).
7. I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, Chapter 7. McGraw-Hill, New York (1956).
8. W. Ritz, Über eine neue Methode zur Lösung Gewisser Variations-probleme der Mathematischen Physik, *J. Reine Angew. Math.* **135**, 1–61 (1908).
9. B. A. Finlayson and L. E. Scriven, The method of weighted residuals—a review, *Appl. Mech. Rev.* **19**, 735–748 (1966).
10. B. A. Finlayson, Existence of variational principles for the Navier–Stokes equations, *Physics Fluids* **15**, 963–967 (1972).
11. H. H. Bossel, Boundary layer computation by an N parameter integral method using exponentials, *AIAA Jl* **8**, 1841–1845 (1970).
12. D. A. MacDonald, Solution of the incompressible boundary layer equations via the Galerkin Kantorovich technique, *J. Inst. Maths Applics* **6**, 115–130 (1970).
13. R. S. Schechter, *The Variational Method in Engineering*. McGraw-Hill, New York (1967).
14. R. S. Schechter, Variational principles for continuum systems, in *Non-equilibrium Thermodynamics, Variational Techniques and Stability*, edited by R. J. Donnelly, R. Herman and I. Prigogine, pp. 55–69. Univ. Chicago Press, Chicago (1966).
15. L. Rosenhead (Ed.), *Laminar Boundary Layers*, Chapter 6. Oxford University Press, Oxford (1963).
16. P. Rosen, Use of restricted variational principles for the solution of differential equations, *J. Appl. Phys.* **25**, 336–338 (1954).
17. D. F. Hays, Variational formulation of the heat equation: Temperature dependent thermal conductivity, in *Non-equilibrium Thermodynamics, Variational Techniques and Stability*, edited by R. J. Donnelly, R. Herman and I. Prigogine, pp. 17–43. Univ. of Chicago Press, Chicago (1966).
18. B. Vujanovic, An approach to linear and nonlinear heat transfer problems using a Lagrangian, *AIAA Jl* **9**, 131–134 (1971).
19. M. Chester, Second sound in solids, *Phys. Rev.* **131**, 2013–2015 (1963).
20. B. Vujanovic and A. M. Strauss, Heat transfer with nonlinear boundary conditions via a variational principle, *AIAA Jl* **9**, 327–330 (1971).
21. T. R. Goodman, Application of integral methods to transient nonlinear heat transfer, in *Advances in Heat Transfer*, Vol. 1, edited by T. F. Irvine, Jr. and J. P. Hartnett, pp. 51–122. Academic Press, New York (1964).

SUR LA METHODE DE POTENTIEL LOCAL APPLIQUEE A LA RESOLUTION DES EQUATIONS DE DIFFUSION

Résumé—On examine de manière critique la méthode du potentiel local en tant qu'outil pour la résolution approchée des équations régissant la diffusion de chaleur et la vorticité.

On en déduit des conditions suffisantes pour établir une correspondance exacte entre le potentiel local et les solutions de multi-moment des équations de diffusion; il n'a pas été trouvé de formulation du potentiel local des équations de diffusion qui ne satisfasse pas ces conditions données.

DIE METHODE DES ÖRTLICHEN POTENTIALS ZUR LÖSUNG VON DIFFUSIONSGLEICHUNGEN

Zusammenfassung—Als Hilfsmittel zur näherungsweisen Lösung von Gleichungen der Turbulenzausbreitung wurde die Methode des örtlichen Potentials untersucht. Ausreichende Bedingungen wurden abgeleitet, um eine genaue Übereinstimmung zwischen örtlichen Potential- und Multimomentlösungen für die Diffusionsgleichungen aufzustellen. Man fand keine Formulierung des örtlichen Potentials für die Diffusionsgleichung, die den gegebenen Bedingungen nicht genügten.

О ПРИМЕНЕНИИ МЕТОДА ЛОКАЛЬНОГО ПОТЕНЦИАЛА ДЛЯ РЕШЕНИЯ УРАВНЕНИЙ ДИФФУЗИИ

Аннотация — Исследуется возможность применения метода локального потенциала для приближенного решения уравнений диффузии.

Найдены достаточные условия для установления точного соответствия между решениями уравнений диффузии, полученными методом локального потенциала и методом моментов. Не было найдено таких решений уравнений диффузии, полученных методом локального потенциала, которые не удовлетворяли бы данным условиям.